Extremal Combinatorics: homework #2*
Due 24 February 2026, at 10pm

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

1. Suppose that H is a graph with $\chi(H) = 3$. Prove that there is a constant $\varepsilon = \varepsilon(H) > 0$ such that $\operatorname{ex}(n, H) \leq \frac{1}{4}n^2 + O(n^{2-\varepsilon})$.

2. Find a graph H with $\chi(H) = 3$ and $\operatorname{ex}(n, H) \geq \frac{1}{4}n^2 + n^{1.99}$ for all large n.

3. Show that there exists a 3-regular n-vertex graph of girth $\Omega(\log n)$ for all large even n.

4. Call a subset X of abelian group Γ an *(s,t)-set* if it is of the form $X = A + B$ for some $A, B \subset \Gamma$ of sizes $|A| = s$ and $|B| = t$. Furthermore, call it *proper (s,t)-set* if $|X| = st$.

   (a) Show that there is function $f_s(t)$ satisfying $\lim_{t\to\infty} f_s(t) = \infty$ such that every (s,t)-set contains a proper $(s, f(t))$-set.

   (b) Show that there is a function $t(s)$ such that, whenever $p \geq p_0(s)$ is a sufficiently large prime, there is a set $S \subset \mathbb{F}_p^s$ of size $|S| \geq \frac{1}{2}p^{s-1}$ that contains no $(s, t(s))$-set.

   (c) Deduce that, for all sufficiently large $N \geq N_0(s)$, the set $[N]$ contains a $(s, t(s))$-good subset of size $\Omega_s(N^{1-1/s})$.

5. Let n be a power of 2. An subinterval of $[n]$ is called *dyadic* if it is of the form $(a2^k, (a+1)2^k]$ for some nonnegative integers a, k. Let $\mathcal{I}_{\text{all}}$ denote the set of all dyadic intervals, and by $\mathcal{I} = \{[2n]\} \cup \{I \in \mathcal{I}_{\text{all}} : |I| \geq 2\}$. Each $I \in \mathcal{I}$ can be split into two equal halves, which are themselves dyadic intervals; we denote

*This homework is from `http://www.borisbukh.org/ExtremalCombinatorics26/hw2.pdf`.

them by left(I) and right(I). For dyadic interval $I \in \mathcal{I}'$ define function

$$b_I(x) = \begin{cases} 1/\sqrt{|I|} & \text{if } x \in \text{left}(I), \\ -1/\sqrt{|I|} & \text{if } x \in \text{right}(I), \\ 0 & \text{if } x \notin I. \end{cases}$$

For a function $f\colon [n] \to \mathbb{R}$ define

$$\widetilde{f}(I) \stackrel{\text{def}}{=} \sum_{x=0}^{n-1} f(x) b_I(x).$$

(a) Prove that, for any $w\colon [n] \to [0,1]$ we have $\sum_{I \in \mathcal{I}} \widetilde{w}(I)^2 \leq n$.

(b) You find yourself playing a strange casino game: The dealer repeatedly tosses a coin. At any time you can make a bet of the form "The fraction of heads among the next R tosses will be approximately ρ." You win the bet if ρ is within 0.1 of the correct fraction. You can place as many bets as you like, and each successful bet pays double of the wager. You suspect that the game is rigged. Devise a strategy that wins over the long term. [Hint: you can use part (a) or ignore it and use something that we learned in class instead.]

(c) (Optional) Can you find both hinted solutions to part (b)?