Turán numbers of theta graphs

Boris Bukh

July 2018

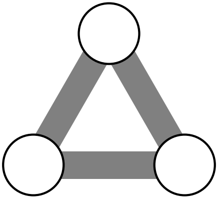
Turán numbers

Forbidden subgraph F. How to make large F-free graph?

$$\mathrm{ex}(n, F) = \max_{\substack{G \text{ is } F\text{-free} \\ n \text{ vertices}}} e(G)$$

Erdős–Stone'46

$$\mathrm{ex}(n, F) = \left(1 - \frac{1}{\chi(F) - 1} + o(1)\right) \binom{n}{2}$$



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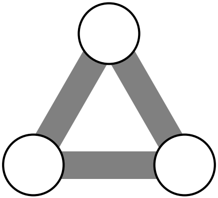

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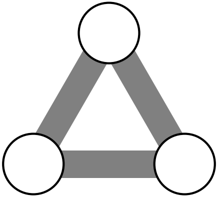
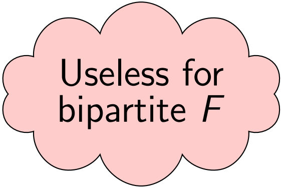




Turán numbers for bipartite graphs: big picture

Complete bipartite graphs:

$\mathrm{ex}(n, K_{2,2}) \sim n^{3/2}$
$\mathrm{ex}(n, K_{3,3}) \sim n^{2-1/3}$
$\mathrm{ex}(n, K_{s,t}) \sim n^{2-1/t}$ if $s > (t-1)!$

Cycles:

$\mathrm{ex}(n, C_{2\ell}) \leq c_\ell n^{1+1/\ell}$ sharp for $\ell = 2, 3, 5$

Other known $\mathrm{ex}(n, F)$ are similar

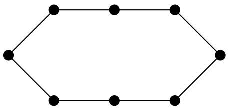
Upper bounds:

Pigeonhole
Easy to challenging

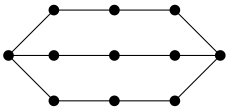
Constructions:

Algebraic graphs
Very hard

Theta graphs

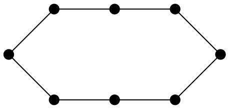


Theta graph $\Theta_{4,2} = C_8$



Theta graph $\Theta_{4,3}$

Theta graphs



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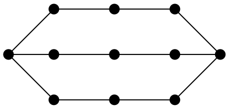

Theta graph $\Theta_{4,3}$

$$\mathrm{ex}(n, C_{2\ell}) \leq \mathrm{ex}(n, \Theta_{\ell,t})$$

Upper bounds:
Harder for $\Theta_{\ell,t}$

Constructions:
Easier for $\Theta_{\ell,t}$

Theta Graphs

Faudree–Simonovits'83:

$$\mathrm{ex}(n, \Theta_{\ell,t}) \leq c_{\ell,t} n^{1+1/\ell} \qquad \text{where } c_{\ell,t} = c_\ell t^{\ell^2}$$

Conlon'14:

$$\mathrm{ex}(n, \Theta_{\ell,t}) \geq \frac{1}{2} n^{1+1/\ell} \qquad \text{for } t \geq t(\ell)$$

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Theorem (B.–Tait'18)

- *For any ℓ, we have* $\mathrm{ex}(n, \Theta_{\ell,t}) \leq c_\ell t^{1-1/\ell} \cdot n^{1+1/\ell}$
- *For odd ℓ, we have* $\mathrm{ex}(n, \Theta_{\ell,t}) \geq c'_\ell t^{1-1/\ell} \cdot n^{1+1/\ell}$

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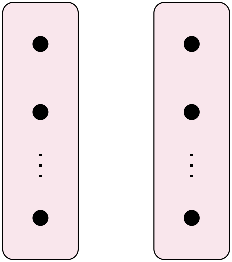
- *For even* ℓ, *we have* $\mathrm{ex}(n, \Theta_{\ell,t}) \geq c'_\ell t^{1/\ell} \cdot n^{1+1/\ell}$

Lower Bounds

Random algebraic constructions:

$K_{s,t(s)}$-free	$n^{2-1/s}$ edges	Blagojević–B.-Karasev'11
$K_{s,t(s)}$-free	$n^{2-1/s}$ edges	B.'14
$\Theta_{\ell,t(\ell)}$-free	$n^{1+1/t}$ edges	Conlon'14

Blowing up Conlon:



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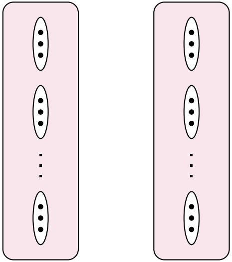

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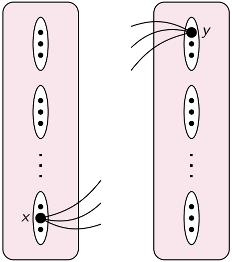

Consider $\Theta_{\ell,T}$:
Endpoints x, y

Lower bounds

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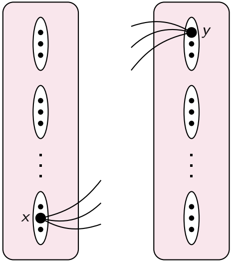


Consider $\Theta_{\ell,T}$:

Endpoints x, y

Key observation:

x, y are in different blobs because ℓ is odd

Conclusion:

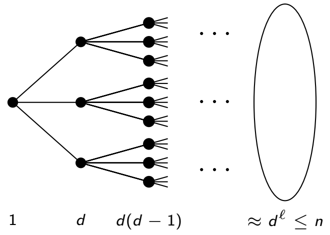
$\Theta_{\ell,T/c}$ in original

Upper Bounds

Easy result:

$\mathrm{ex}(n, \{C_3, C_4, \ldots, C_{2\ell}\}) \leq n^{1+1/\ell}$

Easy proof:

1. G contains no $C_3, C_4, \ldots, C_{2\ell}$
2. Without much loss, G is regular
3. 

Easy result:

$\mathrm{ex}(n, \{C_3, C_4, \ldots, C_{2\ell}\}) \leq n^{1+1/\ell}$

What about $\frac{1}{4} n^{1+1/\ell}$?

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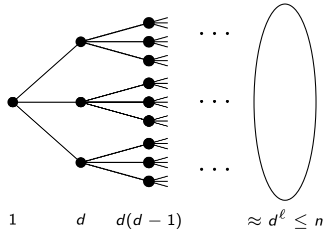
3. 

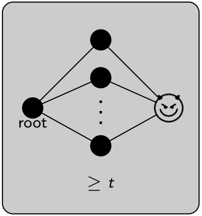
$100 prize

Upper Bounds

Easiest new case: $\text{ex}(n, \Theta_{3,t}) \leq ct^{2/3}n^{4/3}$

Proof:

1. Without much loss, G is d-regular and bipartite



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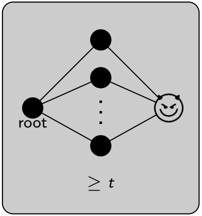

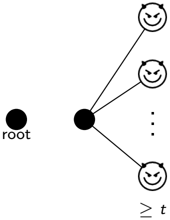

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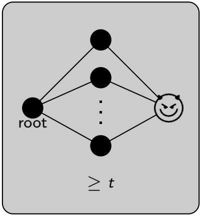

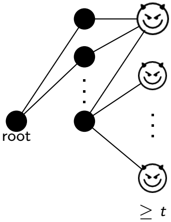

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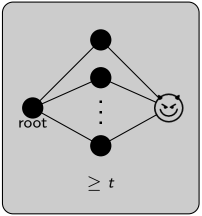

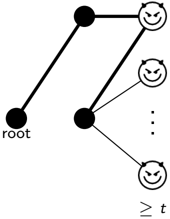

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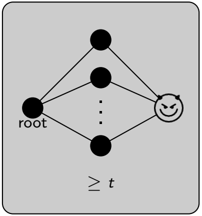

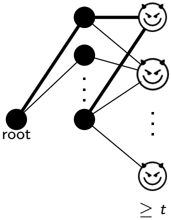

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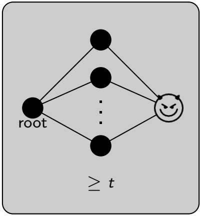

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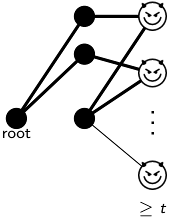

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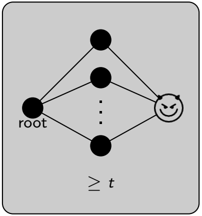

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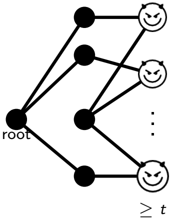

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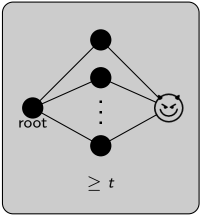

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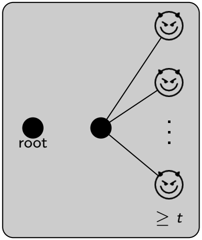

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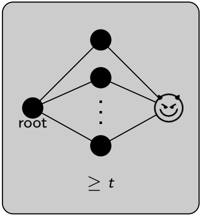

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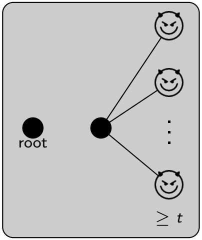

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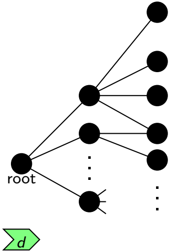

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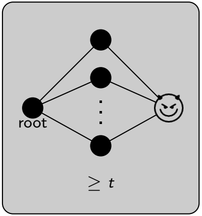

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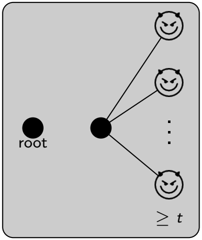

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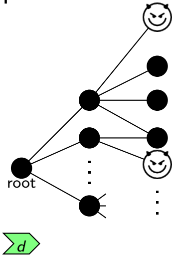

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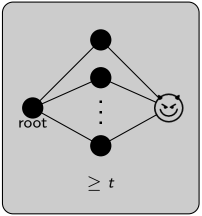

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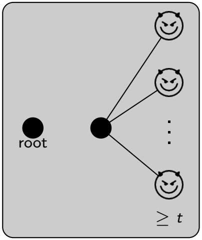

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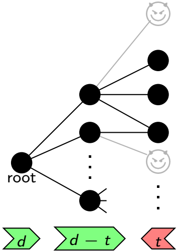

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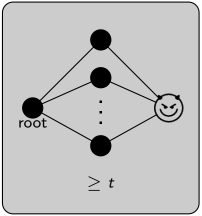

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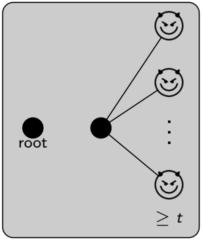

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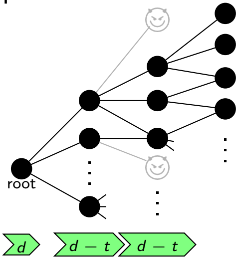

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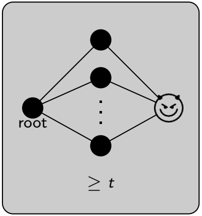

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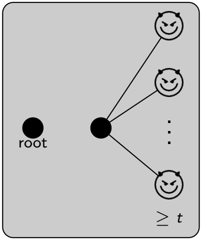

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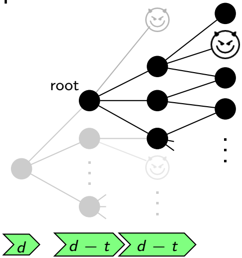

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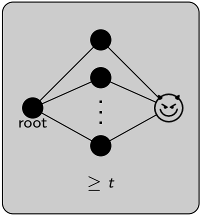

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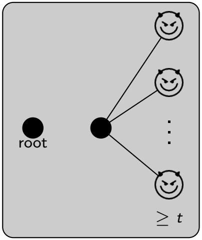
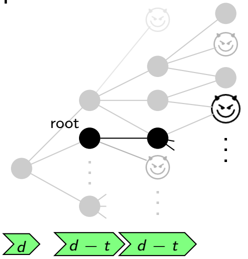


3. Does not occur



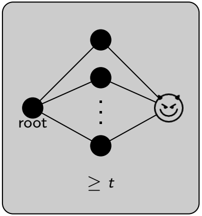
4. Grow tree

Upper Bounds

Easiest new case: $\mathrm{ex}(n, \Theta_{3,t}) \leq ct^{2/3}n^{4/3}$

Proof:

1. Without much loss, G is d-regular and bipartite



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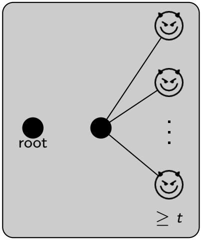

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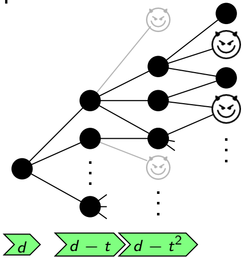

4. Grow tree

Upper Bounds

Easiest new case: $\mathrm{ex}(n, \Theta_{3,t}) \leq ct^{2/3}n^{4/3}$

Proof:

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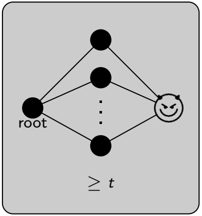

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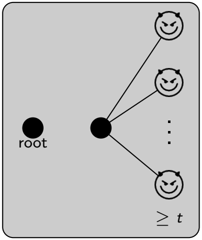

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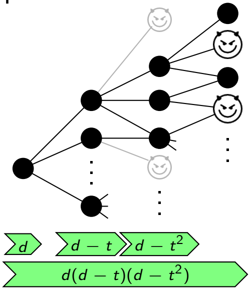


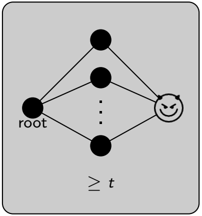
4 Grow tree

Upper Bounds

Easiest new case: $\mathrm{ex}(n, \Theta_{3,t}) \le c t^{2/3} n^{4/3}$

Proof:

1. Without much loss, G is d-regular and bipartite



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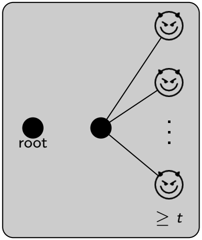

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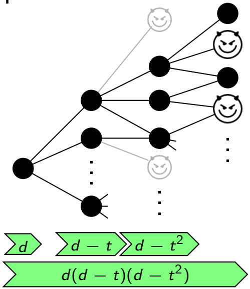
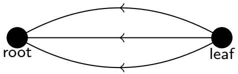


4. Grow tree



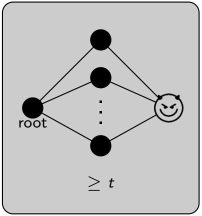
5. Suppose $2t^2$ paths back

Upper Bounds

Easiest new case: $\mathrm{ex}(n, \Theta_{3,t}) \leq ct^{2/3}n^{4/3}$

Proof:

1 Without much loss, G is d-regular and bipartite



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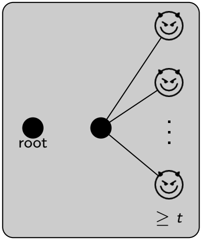

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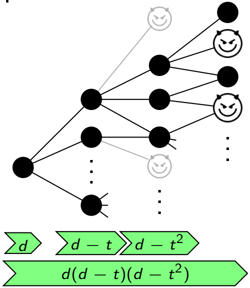
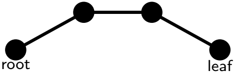


4 Grow tree



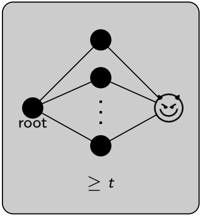
5 Suppose $2t^2$ paths back

Upper Bounds

Easiest new case: $\mathrm{ex}(n, \Theta_{3,t}) \le ct^{2/3} n^{4/3}$

Proof:

1 Without much loss, G is d-regular and bipartite



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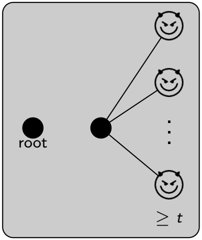

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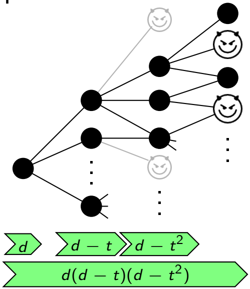
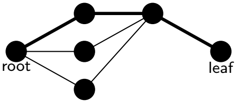


4 Grow tree



5 Suppose $2t^2$ paths back

Upper Bounds

Easiest new case: $\mathrm{ex}(n, \Theta_{3,t}) \leq ct^{2/3} n^{4/3}$

Proof:

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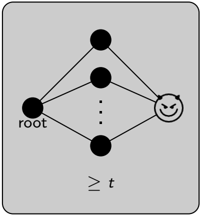

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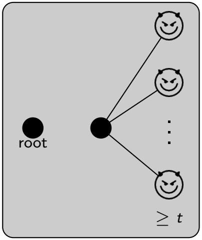

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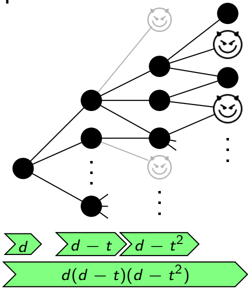

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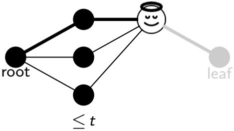

5. Suppose $2t^2$ paths back

Easiest new case: $\mathrm{ex}(n, \Theta_{3,t}) \leq ct^{2/3}n^{4/3}$

Proof:

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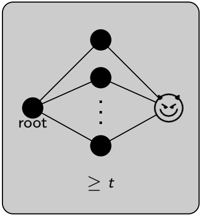

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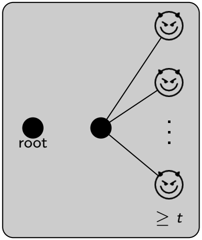
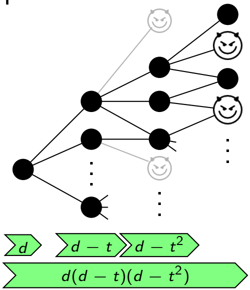
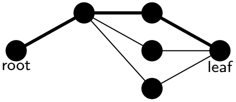


3. Does not occur



4. Grow tree



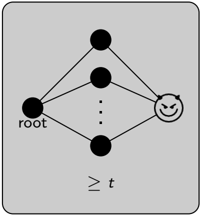
5. Suppose $2t^2$ paths back

Upper Bounds

Easiest new case: $\mathrm{ex}(n, \Theta_{3,t}) \le ct^{2/3}n^{4/3}$

Proof:

1 Without much loss, G is d-regular and bipartite



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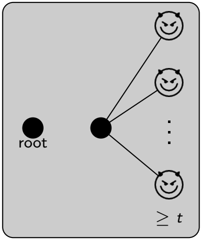
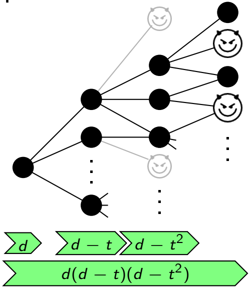
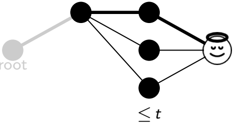


3 Does not occur



4 Grow tree



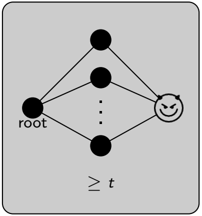
5 Suppose $2t^2$ paths back

Upper Bounds

Easiest new case: $\text{ex}(n, \Theta_{3,t}) \leq ct^{2/3}n^{4/3}$

Proof:

1 Without much loss, G is d-regular and bipartite



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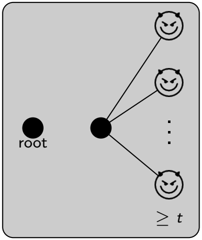

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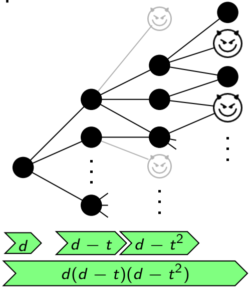
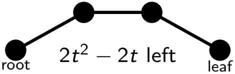


4 Grow tree



5 Suppose $2t^2$ paths back

Upper Bounds

Easiest new case: $\text{ex}(n, \Theta_{3,t}) \leq ct^{2/3}n^{4/3}$

Proof:

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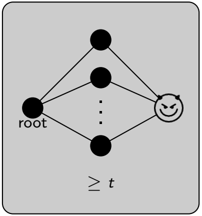

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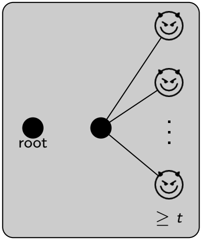

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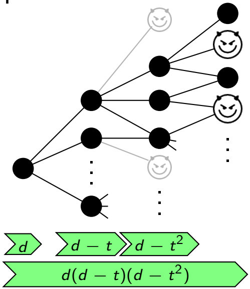
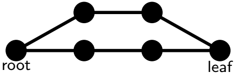


4. Grow tree



$2t^2 - 4t$ left

5. Suppose $2t^2$ paths back

Upper Bounds

Easiest new case: $\mathrm{ex}(n, \Theta_{3,t}) \leq ct^{2/3}n^{4/3}$

Proof:

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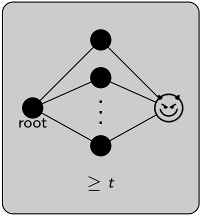

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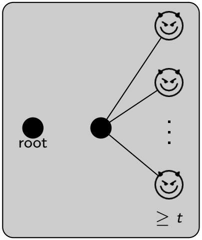
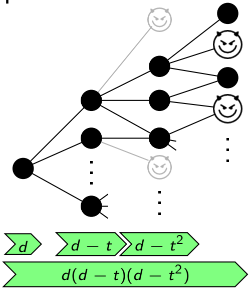
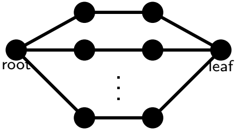


3 Does not occur



4 Grow tree



5 Suppose $2t^2$ paths back

Upper bounds

Easiest new case: $\mathrm{ex}(n, \Theta_{3,t}) \leq ct^{2/3}n^{4/3}$

Proof:

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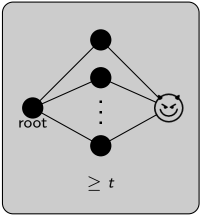

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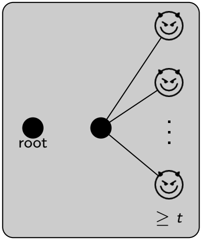

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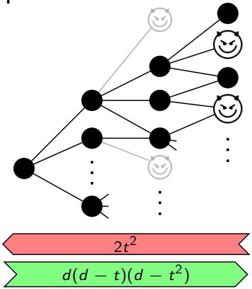

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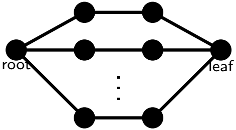

5 Suppose $2t^2$ paths back

$$d(d-t)(d-t^2) \leq 2t^2 \cdot n$$

6 QED

That is all

Now what about $\mathrm{ex}(n, \{C_3, \ldots, C_{2\ell}\})$?