Small antipodal spherical codes

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Joint with Chris Cox

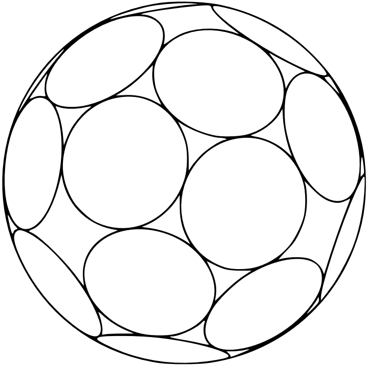
Spherical codes

Spherical code

unit vectors $v_1, v_2, \ldots, v_n$ in $\mathbb{R}^d$

Inner product

$\max_{i \neq j} \langle v_i, v_j \rangle$



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Wanted: large n and small inner product

Special case: binary codes, $v_i \in \{\pm 1/\sqrt{d}\}^d$

Antipodal codes

Antipodal code

vectors come in pairs $v, -v$

Inner product

$\max_{i \neq j} |\langle v_i, v_j \rangle|$

Basic problem

Unit vectors $v_1, v_2, \ldots, v_{d+k} \in \mathbb{R}^d$,

$$f(d,k) = \min_{v_1,\ldots,v_{d+k}} \max_{i \neq j} |\langle v_i, v_j \rangle|$$

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Easy results:

$f(d,0) = 0$ basis

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$f(d,1) = \frac{1}{d}$	simplex

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$f(d,1) = \frac{1}{d}$	simplex
$f(d,2) \lesssim \frac{2}{d}$	simplex $\oplus$ simplex in $\mathbb{R}^{d/2} \oplus \mathbb{R}^{d/2}$

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$f(d,k) \lesssim \frac{k}{d}$	simplex $\oplus \ldots$ in $\mathbb{R}^{d/k} \oplus \ldots$
$f(d,k) \gtrsim \frac{\sqrt{k}}{d}$	using $\operatorname{tr}(A^2) \geq \operatorname{tr}(A)^2 / \operatorname{rk}(A)$ (Welch bound; also LP bound)

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New results:

$f(d,2) \approx \frac{3/2}{d}$

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$f(d,k) \gtrsim \frac{c_k}{d}$ with $c_k = \frac{k(k+1)}{(k-1)\sqrt{k+2}+2}$

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sharp for $k = 0, 1, 2, 3, 7, 23$

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$f(d,k) \leq (1+\varepsilon)\frac{\sqrt{k}}{d}$ where $\varepsilon \to 0$ as $k \to \infty$

Even sharper results for unit vectors in $\mathbb{C}^d$.

No linear programming

Isotropic measures

Isotropic measure μ on $\mathbb{R}^k$

$$\mathbb{E}_{x\sim\mu}|\langle x, v\rangle|^2 = \frac{1}{k}\|v\|^2 \qquad \text{for all vectors } v$$

Key lemma

If μ is isotropic on $\mathbb{R}^k$, then

$$\mathbb{E}_{x,y\sim\mu}|\langle x, y\rangle| \leq \frac{(k-1)\sqrt{k+2}+2}{k(k+1)}$$

Proof for the case $k = 1$

Unit vectors $v_1, \dots, v_{d+1}$ such that $|\langle v_i, v_j \rangle| \leq \varepsilon$

Gram matrix $M = (\langle v_i, v_j \rangle)_{i,j}$

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$$M = \begin{pmatrix} 1 & & & [-\varepsilon, \varepsilon] & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & [-\varepsilon, \varepsilon] & & & 1 & \\ & & & & & 1 \end{pmatrix}$$

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Linear relation $0 = \sum \alpha_i r_i$ wlog $|\alpha_1| \geq |\alpha_2| \geq \cdots \geq |\alpha_{d+1}|$

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Linear relation $0 = \sum \alpha_i r_i$ wlog $|\alpha_1| \geq |\alpha_2| \geq \cdots \geq |\alpha_{d+1}|$

1st column $\alpha_1 = -\sum_{i \geq 2} \alpha_i r_{i1}$

$$\implies |\alpha_1| \leq \varepsilon \sum_{i \geq 2} |\alpha_i| \leq d\varepsilon |\alpha_1|$$

Proof for the case $k = 2$

$$\begin{pmatrix} 1 & & & [-\varepsilon, \varepsilon] & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & [-\varepsilon, \varepsilon] & & & 1 & \\ & & & & & 1 \end{pmatrix}$$

Relation 1	Relation 2
α_1	β_1
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α_3	β_3
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α_1	β_1	$A\alpha_1 + B\beta_1$
α_2	β_2	$A\alpha_2 + B\beta_2$
α_3	β_3	$A\alpha_3 + B\beta_3$
$\vdots$	$\vdots$	$\vdots$

Seek (A, B) such that

$$|A\alpha_1 + B\beta_1| \geq \frac{3}{2}\mathbb{E}_i|A\alpha_i + B\beta_i|$$

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Points $p_i = (\alpha_i, \beta_i)$. Seek $L(x, y) = Ax + By$ such that

$$|Lp_1| \geq \frac{3}{2}\mathbb{E}_i|Lp_i|$$

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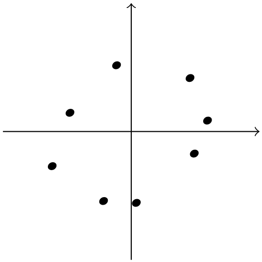

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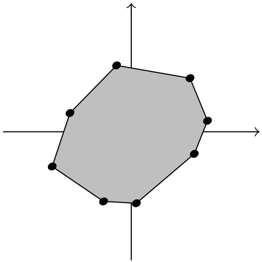

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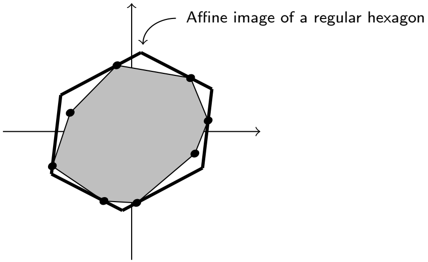

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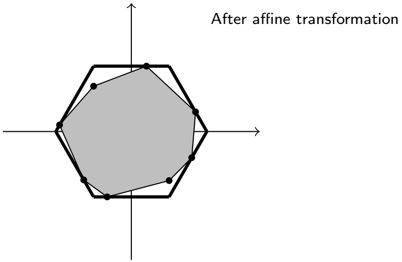

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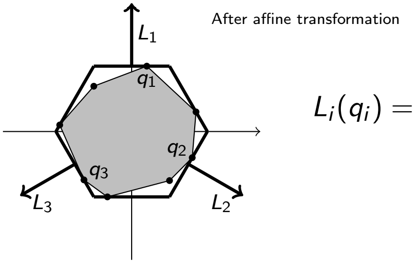

$$L_i(q_i) = 1$$

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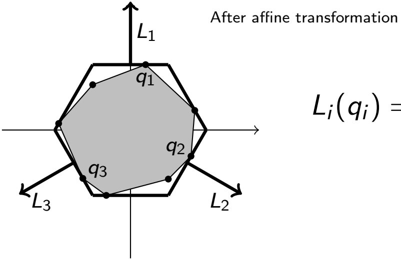


$$L_i(q_i) = 1$$

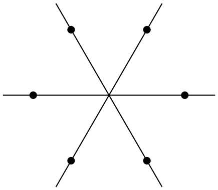
$$\forall p \in \text{⬡} \; |L_1(p)| + |L_2(p)| + |L_3(p)| \leq 2$$

$$\implies \mathbb{E}_i|L_1(p_i)| + |L_2(p_i)| + |L_3(p_i)| \leq 2$$

$$\implies \mathbb{E}_i|L_j(p_i)| \leq 2/3 \text{ for some } j$$

Construction for special k

$k = 2$:



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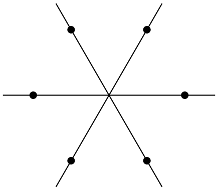


Equiangular lines

every angle is the same

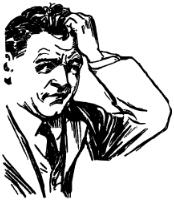
In $\mathbb{R}^k$ at most $\binom{k+1}{2}$ equiangular lines

equality for $k = 0, 1, 2, 3, 7, 23$

Problems

- ℓ^p norm of $|\langle v_i, v_j\rangle|_{i\neq j}$ (this result is $p=\infty$)
  $\implies$ ℓ^q norm of isotropic measures for $\frac{1}{p}+\frac{1}{q}=1$

- Non-antipodal codes (2d+k points)



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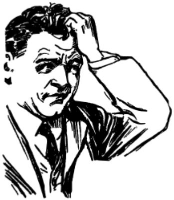
  10hrs ago: Alexey Glazyrin announced a solution

- Non-antipodal codes (2d+k points)

  $\Omega(1/d)$ bound (w/Igor Balla)



Well, you asked for it!

Lemma

If μ is isotropic on $\mathbb{R}^k$, then

$$\mathbb{E}_{x,y\sim\mu}|\langle x, y\rangle| \leq \frac{(k-1)\sqrt{k+2}+2}{k(k+1)}$$

Key quantity:

$$\mathbb{E}_{x,y}\left(\frac{|\langle x, y\rangle|}{\sqrt{\|x\|\|y\|}} - \beta\sqrt{\|x\|\|y\|}\right)^2$$

Upper bound:

1) Expand
2) Cauchy-Schwarz

Lower bound:

1) Cauchy–Schwarz
2) $\operatorname{tr}(A^2) \geq \operatorname{tr}(A)^2/\operatorname{rk}(A)$