Errata for "Multidimensional Kruskal–Katona theorem"

Boris Bukh

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In the paper, it is asserted that the equality in Theorem 1 holds only if $\mathcal{F}$ is of the form $\binom{Y_1}{r} \times \cdots \times \binom{Y_d}{r}$ for some sets $Y_1, \ldots, Y_d \subset X$. The claim is valid, but the proof is not.

The error: A wrong claim appears in the introduction. It is claimed that "...equality holds only if $\mathcal{F}$ is an initial segment of a such colexicographical order". This is false, as shown independently by Füredi–Griggs[2] and by Mörs[5]. Those papers give a thorough treatment to the problem of uniqueness of the extremal family in the Kruskal–Katona theorem.

In particular, the claim is true for families of size $\binom{m}{r}$, where m is an integer. A simple proof of this special case can be found in [1, p. 30] (see also [4] and [3]).

The error in the proof of Theorem 1 lies in application of Lemma 2. The lemma does not assert that if the family $\mathcal{F}_0$ is not monotone, then $|\partial\mathcal{F}_0| < |\partial\mathcal{F}|$, but it is this claim that is implicitly used in the proof of uniqueness in Theorem 1.

A fix: A way to prove the faulty assertion in Theorem 1 is as follows. The application of Lemma 2 to a family $\mathcal{F}$ furnishes a monotone family $\mathcal{F}_0$ that satisfies $|\partial\mathcal{F}_0| \le |\partial\mathcal{F}|$. It is then (correctly) shown that $|\partial\mathcal{F}_0| \ge |LL_r(M_0)|$ where M_0 is the cube of the volume $|\mathcal{F}|$, and that the equality holds only if $\mathcal{F}_0$ is a cube. If $\mathcal{F}_0$ is a cube, then each of its fibers has size that is of the form $\binom{x}{r}$, for same value of x. Since $LL_r(\binom{x}{r}) < KK_r(\binom{x}{r})$ whenever x is not an integer, we may assume that $x \in \mathbb{N}$. As remarked above, if $x \in \mathbb{N}$, then the uniqueness of extremal family holds, and we conclude that the family to which we applied Lemma 2 of $\mathcal{F}$ was monotone after all.

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References

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[4] László Lovász. *Combinatorial problems and exercises.* North-Holland Publishing Co., Amsterdam, second edition, 1993.

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