Ranks of matrices with few distinct entries

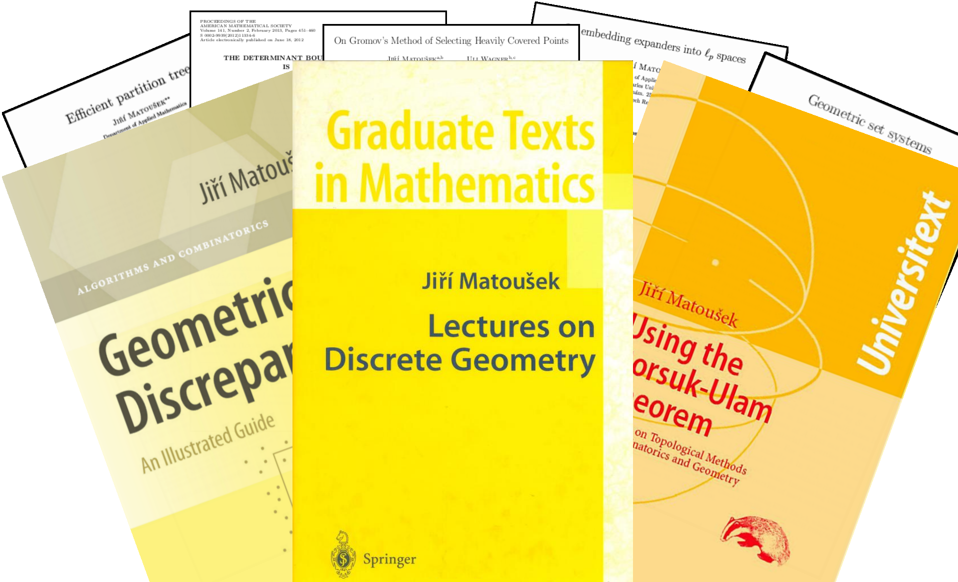
Boris Bukh

July 2016

In memory of Jirka Matoušek

$$\operatorname{rank}\begin{pmatrix} d & & & \in L & & \\ & d & & & & \\ & & d & & & \\ & & & d & & \\ & \in L & & & d & \\ & & & & & d \end{pmatrix}$$

I miss Jirka


Efficient partition trees
On Gromov's Method of Selecting Heavily Covered Points
embedding expanders into ℓ_p spaces
Geometric set systems
Jiří Matoušek
Algorithms and Combinatorics
An Illustrated Guide
Graduate Texts in Mathematics
Jiří Matoušek
Lectures on Discrete Geometry
Springer
Jiří Matoušek
Universitext

Jirka and notation

1 June 2012

I am writing a draft

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into~$\seq a$. Quantitatively, it suffices to take
\[
N:= 2^{R^{2n}},
\]
provided that $R\ge R_0$ for a sufficiently large constant~$R_0$.
\end{prop}

\begin{proof}
Let $n$ and $R$ be given, let $N$ be sufficiently large
as in the proposition, and let $\seq a$ be a sequence of length~$N$.

First, ...

\end{proof}
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Jirka and notation

1 June 2012

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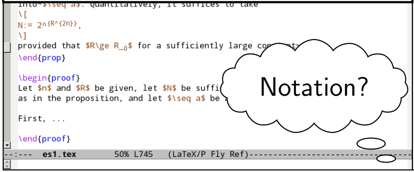

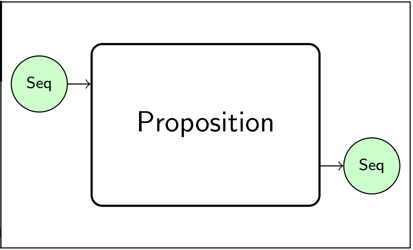

Jirka and notation

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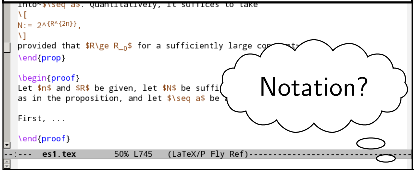

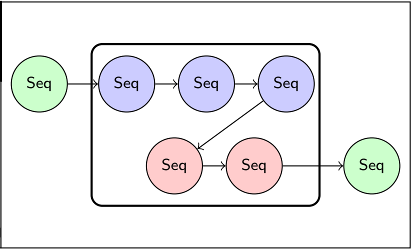

Jirka and notation

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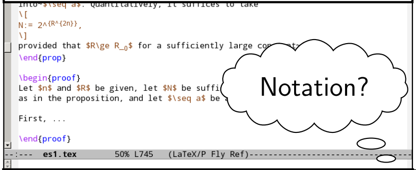
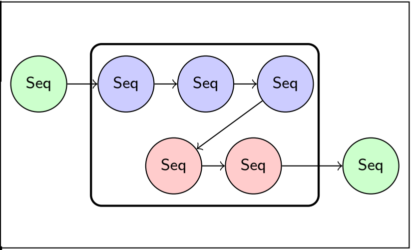




15 minutes of brainstorming

a, b, c, d, e

a_1, a_2, a_3, a_4, a_5

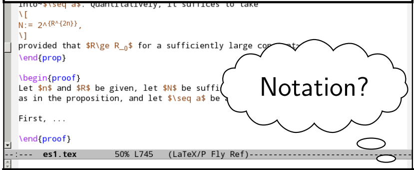
$a^{(1)}, a^{(2)}, a^{(3)}, a^{(4)}, a^{(5)}$

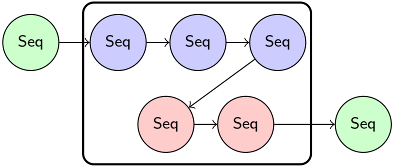
$a^{(1)}, a^{(2)}, a^{(3)}, b^{(1)}, b^{(2)}$

Jirka and notation

1 June 2012

I am writing a draft





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7 June 2012

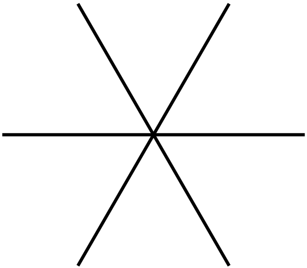
Revision from Jirka

$a^{(1)}, a^{(2)}, a^{(3)}, b^{(3)}, b^{(4)}$

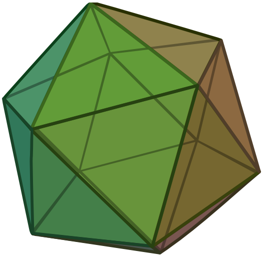
A special case: equiangular lines

Family $\mathcal{L}$ of lines in $\mathbb{R}^d$ is *equiangular* when all pairwise angles $\measuredangle \ell\ell'$ are equal, for $\ell, \ell' \in \mathcal{L}$

Examples:



$d = 2$



$d = 3$
(Large diagonals)

Gram matrices

Lines $l_1, \ldots, L_n$ in $\mathbb{R}^d$

$\Downarrow$

Unit vectors $v_1, \ldots, v_n$ in $\mathbb{R}^d$
(line directions)

$\Downarrow$

Matrix of inner products $\big(\langle v_i, v_j \rangle\big)_{i,j}$
(Gram matrix)

Gram matrices

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Equiangular

$\Downarrow$

????

Gram matrices

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Unit vectors $v_1, \ldots, v_n$ in $\mathbb{R}^d$
(line directions)

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Matrix of inner products $\big(\langle v_i, v_j \rangle\big)_{i,j}$
(Gram matrix)

Equiangular

$\Downarrow$

$\langle v_i, v_j \rangle \in \{-\alpha, +\alpha\}$

$\Downarrow$

$$\begin{pmatrix} 1 & & \pm\alpha \\ & 1 & \\ \pm\alpha & & 1 \\ & & & 1 \end{pmatrix}$$

Gram matrices

Lines $l_1, \dots, L_n$ in $\mathbb{R}^d$ — Equiangular

$\Updownarrow$ $\Updownarrow$

Unit vectors $v_1, \dots, v_n$ in $\mathbb{R}^d$ (line directions) — $\langle v_i, v_j \rangle \in \{-\alpha, +\alpha\}$

$\Updownarrow$ $\Updownarrow$

Matrix of inner products $\big(\langle v_i, v_j \rangle\big)_{i,j}$ (Gram matrix)

Positive semidefinite

$$\begin{pmatrix} 1 & & \pm\alpha \\ & 1 & \\ \pm\alpha & & 1 \\ & & & 1 \end{pmatrix}$$

Gram matrices

Unit vectors $v_1, v_2, \ldots, v_n$ in $\mathbb{R}^d$ — n vectors

$\Downarrow$

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}$$

Rank $\leq d$

$\Downarrow$

Gram matrix $M = A^T A$ — Rank $\leq d$

General problem

How small can a rank of an (L, d)-matrix be?

General (L, d)-matrix

$$\begin{pmatrix} d & & & \in L & & \\ & d & & & & \\ & & d & & & \\ & & & d & & \\ & \in L & & & d & \\ & & & & & d \end{pmatrix}$$

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General (L, d)-matrix

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If M is an (L, d)-matrix, then $M - dJ$ is $(L - d, 0)$-matrix of almost the same rank. So, with little loss we may assume that $d = 0$.

Special case: graph eigenvalues

What is the maximum eigenvalue multiplicity of λ?

Details:

- Number λ is fixed
- We consider adjacency matrices of graphs on n vertices
- We seek the graph that maximizes the multiplicity of eigenvalue λ

Special case: graph eigenvalues

What is the maximum eigenvalue multiplicity of λ?

General adjacency matrix:

:

$$\begin{pmatrix} 0 & & & & & \\ & 0 & & \{0,1\} & & \\ & & 0 & & & \\ & & & 0 & & \\ & \{0,1\} & & & 0 & \\ & & & & & 0 \end{pmatrix}$$

Special case: graph eigenvalues

What is the maximum eigenvalue multiplicity of λ?

Multiplicity of λ in a general adjacency matrix: $\iff$ Nullity of a matrix of the form:

$$\begin{pmatrix} 0 & & & \{0,1\} & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ \{0,1\} & & & & 0 & \\ & & & & & 0 \end{pmatrix} \qquad \begin{pmatrix} -\lambda & & & \{0,1\} & & \\ & -\lambda & & & & \\ & & -\lambda & & & \\ & & & -\lambda & & \\ \{0,1\} & & & & -\lambda & \\ & & & & & -\lambda \end{pmatrix}$$

Rank + nullity = n

(L, d)-matrices: some examples

- Equiangular lines
- Multiplicity of graph eigenvalues
- Sets in $\mathbb{R}^d$ with few distances
- Set systems with restricted intersection

(L, d)-matrices: some examples

- Equiangular lines
- Multiplicity of graph eigenvalues
- Sets in $\mathbb{R}^d$ with few distances
- Set systems with restricted intersection

  $S_1, \ldots, S_n$ are d-element sets with $|S_i \cap S_j| \in L$

  $v_1, \ldots, v_n$ are characteristic vectors

  $$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix} \text{ is made of 0's and 1's}$$

  $M = A^T A$ is an (L, d)-matrix

L-matrices: the upper bound

General L-matrix

$$\begin{pmatrix} 0 & & & & & \\ & 0 & & \in L & & \\ & & 0 & & & \\ & & & 0 & & \\ & \in L & & & 0 & \\ & & & & & 0 \end{pmatrix}$$

"Polynomial method" (Koornwinder? Frankl–Wilson?)

Suppose $|L| = k$ and $0 \notin L$, and M is an n-by-n L-matrix of rank r. Then

$$n \leq \binom{r+k}{k}.$$

L-matrices: the upper bound

General L-matrix

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Sharp for some sets L

An example

$$\begin{pmatrix} 0 & & & & & \\ & 0 & & & \{1,3\} & \\ & & 0 & & & \\ & & & 0 & & \\ & \{1,3\} & & & 0 & \\ & & & & & 0 \end{pmatrix}$$

Polynomial method: rank $r \implies$ size at most $O(r^2)$

An example

$$\begin{pmatrix} 0 & & & & & \\ & 0 & & \{1,3\} & & \\ & & 0 & & & \\ & & & 0 & & \\ & \{1,3\} & & & 0 & \\ & & & & & 0 \end{pmatrix}$$

Polynomial method: rank $r \implies$ size at most $O(r^2)$

Modulo 2: almost full rank, size at most $r+1$

General results

$$N(r, L) = \max\{n : \text{there is an } n\text{-by-}n\ L\text{-matrix of rank } \leq r\}.$$

Theorem (B.)

For a set $L = \{\alpha_1, \ldots, \alpha_k\}$, the following are equivalent

1. $N(r-1, L) > r$ *for some* r
2. *There is an integer homogeneous polynomial* P *s.t.* $P(\alpha_1, \ldots, \alpha_k) = 0$ *and* $P(1, 1, \ldots, 1) = 1$
3. $\lim_{r\to\infty} N(r, L)/r$ *exists and is* > 1

General results

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For a set $L = \{\alpha_1, \ldots, \alpha_k\}$, the following are equivalent

1. $N(r-1, L) > kr$ *for some* r
2. *There is a* ~~a~~ linear *integer homogeneous polynomial* P *s.t.* $P(\alpha_1, \ldots, \alpha_k) = 0$ *and* $P(1, 1, \ldots, 1) = 1$
3. $N(r, L) = \Omega(r^{3/2})$

Corollaries for the special case

$$G(n, \lambda) = \max\{\text{mult. } \lambda \text{ in a } n\text{-vertex graph}\}$$

$$D(n, \lambda) = \max\{\text{mult. } \lambda \text{ in a } n\text{-vertex digraph}\}$$

Theorem (B.)

1. *If λ is an algebraic integer of degree d, then*

$$D(n, \lambda) = n/d - O(\sqrt{n}).$$

2. *Otherwise, λ is not an eigenvalue of any $\{0, 1\}$-matrix*

Graph eigenvalues:

Same holds for $G(n, \lambda)$ if degree of λ is at most 4

The general case is open

Proofs: algebraic reason

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Proof of 1 $\implies$ 2.

Assume M is an L-matrix of size n.

Let $P_n(\alpha_1, \ldots, \alpha_n) \stackrel{\text{def}}{=} \det M$, homogeneous of degree n.

$$P_n(\alpha_1, \ldots, \alpha_k) = \det \begin{pmatrix} 0 & \alpha_1 & \cdots & \alpha_3 \\ \alpha_2 & 0 & \cdots & \alpha_1 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1 & \alpha_1 & \cdots & 0 \end{pmatrix}$$

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$$P_n(1, \ldots, 1) = \det \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix} = (-1)^{n-1}(n-1)$$

Proofs: algebraic reason

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For $L = \{\alpha_1, \dots, \alpha_k\}$, the following are equivalent

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Assume M is an L-matrix of size n, M' is a submatrix of size $n-1$

Let $P_n(\alpha_1, \dots, \alpha_n) \stackrel{\text{def}}{=} \det M$, homogeneous of degree n.

Let $P_{n-1}(\alpha_1, \dots, \alpha_n) \stackrel{\text{def}}{=} \det M'$, homogeneous of degree $n-1$.

$$P_n(1, \dots, 1) = (-1)^{n-1}(n-1)$$

$$P_{n-1}(1, \dots, 1) = (-1)^{n-2}(n-2)$$

Proofs: algebraic reason

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$$\left.\begin{aligned} P_n(1, \dots, 1) &= (-1)^{n-1}(n-1) \\ P_{n-1}(1, \dots, 1) &= (-1)^{n-2}(n-2) \end{aligned}\right\} \implies \begin{aligned} &P = (P_n - \alpha_1 P_{n-1})^2 \\ &P(\alpha_1, \dots, \alpha_k) = 0 \quad \square \end{aligned}$$

Proofs: algebraic reason

$N(r, L) = \max\{n : \text{there is an } n\text{-by-}n\ L\text{-matrix of rank } \leq r\}$.

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If $N(r-1, L)$ is large, P vanishes to high order

Proofs: high vanishing lemma

Lemma (B.)

Let $\alpha = (\alpha_1, \ldots, \alpha_k)$. If $P(x_1, \ldots, x_k)$ is an integer homogeneous polynomial such that

1. *P vanishes at α to order $> \frac{k-1}{k} \deg P$,*
2. *$P(1, \ldots, 1) = 1$.*

Then there is a linear polynomial Q such that

1. *Q vanishes at α,*
2. *$Q(1, \ldots, 1) = 1$.*

Case $k = 2$ is a consequence of Gauss's lemma: if $P(x)$ vanishes at α to order $> \frac{1}{2} \deg P$, then a linear factor of P vanishes at α.

General case uses a contagious vanishing argument (Baker, Guth–Katz, etc)

Proofs: digraphs with massive eigenvalues

1. If λ is an algebraic integer of degree d, then

$$D(n, \lambda) = n/d - O(\sqrt{n}).$$

2. Otherwise, λ is not an eigenvalue of any $\{0, 1\}$-matrix

Proof of 2.

- Characteristic polynomial P of a $\{0, 1\}$-matrix is monic with integer coefficients
- Eigenvalues are roots of P, with respective multiplicity
- Let Q be the min. polynomial of λ, then $Q^{\text{mult}\,\lambda}$ divides P. □

Proofs: digraphs with massive eigenvalues

1. If λ is an algebraic integer of degree d, then

$$D(n, \lambda) = n/d - O(\sqrt{n}).$$

Proof of the lower bound in 1 .

- There is a size-d matrix M with integer coefficients such that λ is an eigenvalue (companion matrix)
- Multiplicity of λ in $M \otimes I_\ell$ is ℓ

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Proof of the lower bound in 1.

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- Multiplicity of λ in $M \otimes I_\ell$ is ℓ

$$M \otimes I_\ell = \begin{pmatrix} M_{11} I_\ell & M_{12} I_\ell & \cdots & M_{1d} I_\ell \\ M_{21} I_\ell & M_{22} I_\ell & \cdots & M_{2d} I_\ell \\ \vdots & \vdots & \ddots & \vdots \\ M_{d1} I_\ell & M_{d2} I_\ell & \cdots & M_{dd} I_\ell \end{pmatrix}$$

- Add a matrix of rank $O(\sqrt{\ell})$ to each block, to turn $M \otimes I_\ell$ into a $\{0, 1\}$-matrix. Only d^2 blocks.

Proofs: digraphs with massive eigenvalues

1. If λ is an algebraic integer of degree d, then

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Proof of the lower bound in 1.

- Add a matrix of rank $O(\sqrt{\ell})$ to each block, to turn $M \otimes I_\ell$ into a $\{0,1\}$-matrix. Only d^2 blocks.
- Example: Want to turn $-2I_\ell$ into a $\{0,1\}$-matrix.

$S_1, \dots, S_\ell$ be two-element sets in $\{1, 2, \dots, 2\sqrt{\ell}\}$

$v_1, \dots, v_\ell$ be characteristic vectors

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_\ell \\ | & | & & | \end{pmatrix}$$

$\Delta = A^T A$ is a $(\{0,1\}, 2)$-matrix of rank $\leq 2\sqrt{\ell}$ □

E

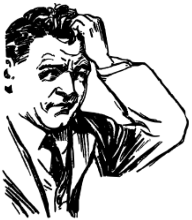
G

Open problems

- Is there a $\{\ell, \ell+1\}$-matrix of rank r and size $\frac{1}{100}r^2$?

- If $\deg \lambda = d$, prove that the maximum multiplicity of λ in a graph is at most $n/d - 100$ for large n.

- What is $N(L, r)$ for a random subset L of $\{1, 2, \ldots, m\}$? (Application: explicit construction of Ramsey graphs)



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Mathematics is Beautiful!

Equiangular lines

$N(d)$ maximum number equiangular lines in $\mathbb{R}^d$

$N_\alpha(d)$ same as $N(d)$, but with $\langle v_i, v_j \rangle \in \{\pm\alpha\}$

Known bounds:

$N(d) \leq d(d+1)/2$ — Polynomial method

$N_\alpha(d) \leq d\frac{1-\alpha^2}{1-d\alpha^2}$ if $d < 1/\alpha^2$ — Nearly identity matrix

$N_\alpha(d) \leq 2d$ if $\alpha \notin \{\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \dots\}$ — Characteristic polynomial

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$N_\alpha(d) \leq 2d$ if $\alpha \notin \{\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \dots\}$ — Characteristic polynomial

$N_{1/(2r-1)}(d) \geq \frac{r}{r-1}d + O(1)$ — Tensor product

$N \geq \frac{2}{9}(d+1)^2 + O(1)$ — Miracle

Equiangular lines

$N_{1/3}(d) = 2d - 2$	for $d \geq 15$	Lemmens–Seidel
$N_{1/5}(d) = \lfloor 3(d-1)/2 \rfloor$	for large d	Neumaier, Greaves–Koolen–Munemasa–Szöllösi

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Theorem (B.)

For a fixed α, the maximum number of equiangular lines satisfies

$$N_\alpha(d) \leq c_\alpha d$$

for some constant c_α.

Equiangular lines

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$N_{1/5}(d) = \lfloor 3(d-1)/2 \rfloor$	for large d	Neumaier, Greaves–Koolen–Munemasa–Szöllösi

Theorem (B.)

For a fixed α, the maximum number of equiangular lines satisfies

$$N_\alpha(d) \leq c_\alpha d$$

for some constant c_α.

My proof gave a HUGE bound on c_α.

Balla–Dräxler–Keevash–Sudakov have improved this to $c_\alpha \leq 2$.

Equiangular lines: basic idea

Unit vectors $v_1, \ldots, v_n$ form an L-spherical code if

$$\langle v_i, v_j \rangle \in L \quad \text{for distinct } i, j.$$

Equiangular lines form a $\{-\alpha, +\alpha\}$-spherical code.

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Size of any $[-1, -\beta] \cup \{\alpha\}$-spherical code in $\mathbb{R}^d$ is at most $c_\beta d$.

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Basic ingredients:

- A $[-1, -\beta]$-spherical code has at most $1/\beta + 1$ elements
- A $\{\alpha\}$-spherical code has at most d elements
- Ramsey's theorem

Graph:

- Vertices $\{v_1, \ldots, v_n\}$;
- Edges: $v_i v_j$ if $\langle v_i, v_j \rangle \leq -\beta$

No clique of size $1/\beta + 2$

No indep. set of size $d + 1$

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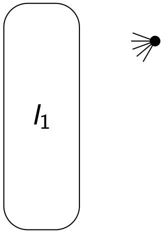
Argument:

- Find a large maximal independent set I_1 (simplex)
- For $v_i \notin I_1$ there must be many edges from v_i to I_1



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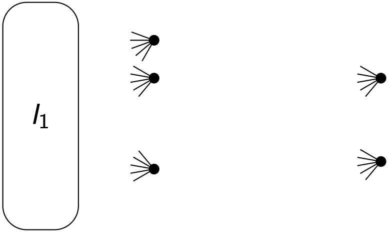

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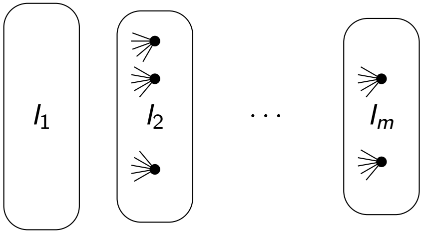

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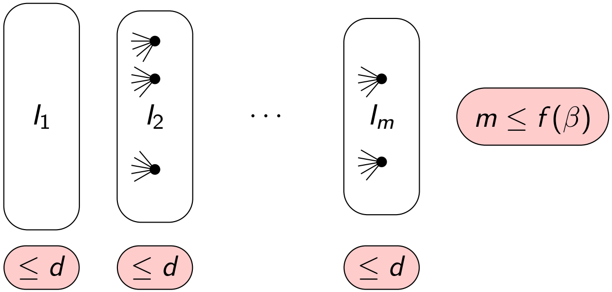

Graph eigenvalue multiplicity

λ is totally real if all of its Galois conjugates are real

Observation

Eigenvalues of a graph are totally real.

Proof.

Eigenvalues of a symmetric real matrix are real. □

So, assume that λ is totally real of degree d.

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However!

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & -1 \end{pmatrix}$$

has eigenvalue $\sqrt{3}$
with multiplicity 2

Graph eigenvalues: representability

Call λ of degree d representable if there is a symmetric size-md matrix in which λ has multiplicity m

Which λ are representable?

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Which λ are representable?

Theorem (Estes–Gularnick)

All totally real algebraic integers of degree $d \leq 4$ are representable.

Theorem

There is a non-representable λ of degree 2880 (Dobrowolski)
There is a non-representable λ of degree 6 (McKee)