Geometric selection theorems

Boris Bukh
joint with Jiří Matoušek and Gabriel Nivasch

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Two theorems

Theorem (Rado'46)

For any set P of n points in $\mathbb{R}^d$ there is a point p (centerpoint) such that $|H \cap P| \geq \frac{1}{d+1}|P|$ for every closed halfspace containing p.

Theorem (Vapnik–Chervonenkis'71)

For any set P of n points in $\mathbb{R}^2$ there is a set N (net) of

$$\frac{200}{\epsilon^2} \log \frac{200}{\epsilon}$$

points such that

$$\left| \frac{|T \cap P|}{|P|} - \frac{|T \cap N|}{|N|} \right| \leq \epsilon \qquad \textit{for every triangle } T.$$

Two theorems and more

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Theorem (Haussler-Welzl'87)

For any set P of n points in $\mathbb{R}^2$ there is a set N (net) of

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points such that

$$\frac{|T \cap P|}{|P|} \geq \epsilon \implies T \cap N \neq \emptyset \qquad \text{for every triangle } T.$$

Introduction

Basic problem

Let S be a large set of points in $\mathbb{R}^d$. Approximate S by a small set N that behaves similarly to S.

Definition

Suppose $S \subset \mathbb{R}^d$ and $\mathcal{F}$ is a family of sets in $\mathbb{R}^d$. Then $N \subset S$ is an *ϵ-net for S* (with respect to $\mathcal{F}$) if N intersects every $F \in \mathcal{F}$ whenever $|F \cap S| \geq \epsilon |S|$.

Wonders of VC-dimension...

Definition

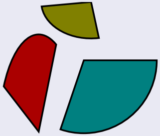
Vapnik-Chervonenkis dimension (abbreviated VC dimension) of a set family $\mathcal{F} \subset 2^X$ is at least d if there is $|Y| = d + 1$, for which $\mathcal{F}|_Y := \{F \cap Y : F \in \mathcal{F}\}$ is the powerset 2^Y.

Theorem

If the family $\mathcal{F}$ has finite VC dimension, then for every S there is always an $1/r$-net whose size $cr \log r$. (No matter how large S is!)

Fact

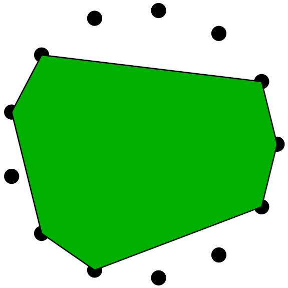
Family of all n-face polyhedra in $\mathbb{R}^d$ has finite VC dimension. More generally, the family of semialgebraic sets of complexity n has finite VC dimension.



. . . and its shortcomings

Fact

The family of all the convex sets has infinite VC dimension. There are no small ϵ-nets for points in convex position.



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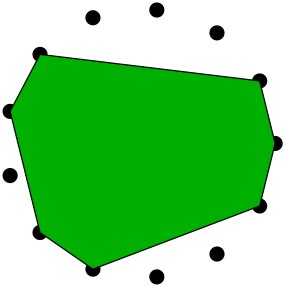

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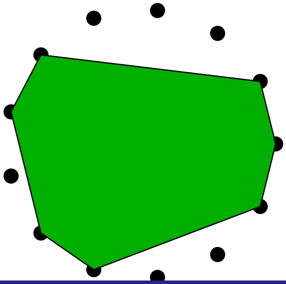

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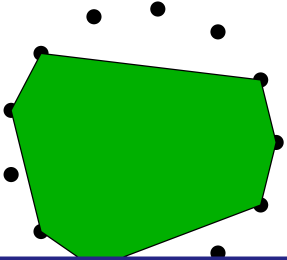

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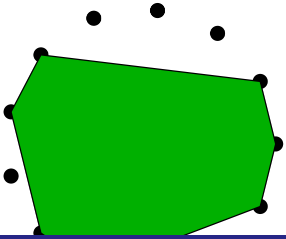

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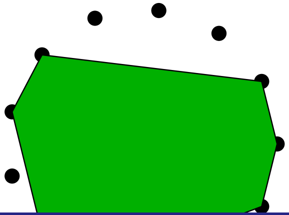

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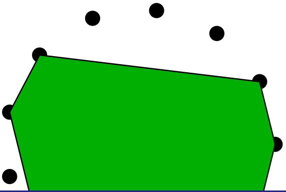

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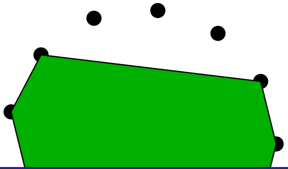

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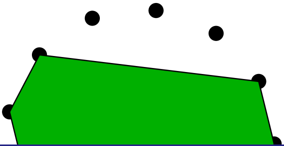

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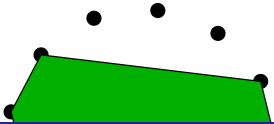

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- Bárány, Füredi, Lovász'90: There are weak $1/r$-nets of size r^{1026} in $\mathbb{R}^2$.
- Alon, Bárány, Füredi, Kleitman'92: There are weak $1/r$-nets of size r^2 in $\mathbb{R}^2$, and of size $r^{d+1-\epsilon(d)}$ in $\mathbb{R}^d$, $d \geq 3$.
- Chazelle, Edelsbrunner, Grigni, Guibas, Sharir, Welzl'95: There are weak $1/r$-nets of size $r^d \log^{c(d)} r$ in $\mathbb{R}^d$.
- Matoušek, Wagner'04: There are weak $1/r$-nets of size $r^d \log^{c(d)} r$ in $\mathbb{R}^d$, with smaller $c(d)$.

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- **Triviality**: There is no weak $1/r$-net of size less than r.
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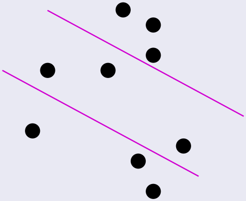
Lower bound on weak ϵ-nets

Triviality

Every weak $1/r$-net for any set $S \subset \mathbb{R}^d$ has at least r points.

Proof.

Partition S into r equal parts by $r-1$ parallel hyperplanes. The slab between every pair of adjacent hyperplanes must contain a point of a weak $1/r$-net. □



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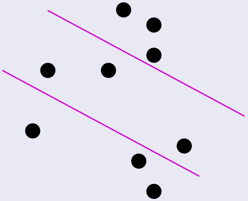

Theorem (B., Matoušek, Nivasch)

There is a set $S \subset \mathbb{R}^d$ for which every weak $1/r$-net has at least $c_d r \log^{d-1} r$ points.

Approximation by a single point

How well can one approximate a set by a *single* point?

Problem

Every set of n points $S \subset \mathbb{R}^d$ determines $\binom{n}{d+1}$ simplices. How many of the simplices can be stabbed by a single point?

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For every $S \subset \mathbb{R}^d$ there is a point that stabs $c_d n^{d+1}$ simplices spanned by S. The constant $c_d > 0$ depends only on the dimension d.

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Bárány'82:	$\frac{1}{d!(d+1)^{d+1}} \leq c_d$.
Boros–Füredi'84:	$\frac{1}{27} \leq c_2 \leq \frac{1}{27} + \frac{1}{729}$.
Wagner'03:	$\frac{d^2+1}{(d+1)!(d+1)^{d+1}} \leq c_d$.
Gromov'?? (draft):	$\frac{2d}{(d+1)(d+1)!^2} \leq c_d$ (in topological setting).

Theorem (B.–Matoušek–Nivasch'08)

There is a construction which demonstrates that $c_d \leq (d+1)^{-(d+1)}$.

Approximation by a single point: sparse case

Problem

Every set of n points $S \subset \mathbb{R}^2$ determines $\binom{n}{3}$ triangles. **Let T be a family of any $\alpha\binom{n}{3}$ of these triangles.** *How many of these triangles can be stabbed (intersected) by a single point?*

Theorem (Eppstein'93)

For every point set S and every family of triangles T, there is a point stabbing at least $c\alpha^3 \operatorname{polylog}(\alpha)\binom{n}{3}$ triangles of T.

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For every point set S there is a family of triangles T, with no point stabbing more than $c\alpha^2\binom{n}{3}$ triangles of T.

Approximation by a single point: sparse case

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Theorem (B.–Matoušek–Nivasch)

There is a point set S and a family of triangles T, with no point stabbing more than $c\frac{\alpha^2}{\log(1/\alpha)}\binom{n}{3}$ triangles of T.

Why are the constructions difficult?

There are many candidates for sets with no small weak ϵ-nets: a chunck of $\mathbb{Z}^d$ lattice, points on a moment curve, points on a sphere, and many others. Probably they all give non-trivial lower bounds, but we cannot prove it.

Main idea

Use any construction whose intersection with convex sets is very simple to describe.

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Conjecture

For no set $S \subset \mathbb{R}^d$, $d \geq 3$, in general position there is a weak $1/r$-net with only $O(r)$ points.

Construction ($d = 2$)

Let $A \ll B$ mean that A is **much** smaller than B. Pick

$$x_1 \ll x_2 \ll \cdots \ll x_n \ll y_1 \ll y_2 \ll \cdots \ll y_n.$$

Let

$$X = \{x_1, \ldots, x_m\}, \qquad Y = \{y_1, \cdots, y_m\}.$$

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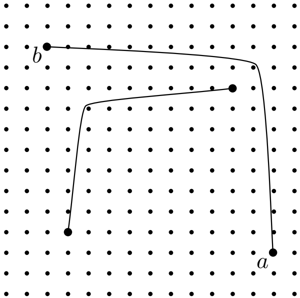
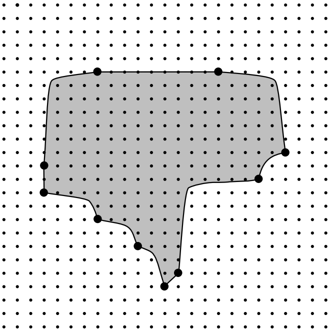


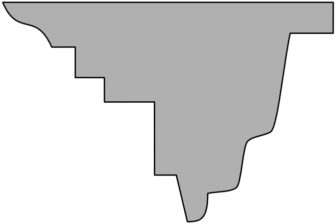
Line segments



Typical convex hull

Main lemma

In the limit the convex sets have flat top envelope, and unimodal bottom envelope. These are called *stairconvex sets*.



Identify the grid $G = X \times Y$ with the grid $\{0, 1/m, \ldots, (m-1)/m\}^2$ inside $[0,1]^2$.

Lemma

Suppose N is a set of n points in $[0,1]^2$. Then there is a stairconvex set $C \subset [0,1]^2$ of area $c\frac{\log n}{n}$ that misses N.