Measurable chromatic number and sets with excluded distances

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D-avoiding sets

Definition

For a set of distances $D = \{d_1, \ldots, d_k\} \subset \mathbb{R}^+$ a set $A \subset \mathbb{R}^2$ is D-avoiding if $x, y \in A$ implies $|x - y| \notin D$.

Definition

Graph G_D has vertex set $\mathbb{R}^2$ and edges $x \sim y$ whenever $|x - y| \in D$.

Observation

Independent set in G_D = D-avoiding set.

Chromatic number: measurable and not

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$\chi(G_{\{1\}})$ – chromatic number of the plane

Theorem (Compactness)

In ZFC: $\chi(G) = \max_{H \subset G} \chi(H)$ *if* $\chi(G) < \infty$.

Theorem (Solovay'70)

ZF+ "all subsets of $\mathbb{R}$ *are measurable" is consistent.*

Chromatic number: measurable and not

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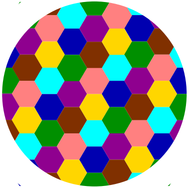
Graph G_D has vertex set $\mathbb{R}^2$ and edges $x \sim y$ whenever $|x - y| \in D$.

Definition

Measurable chromatic number $\chi_m(G_D)$ is the smallest number of measurable D-avoiding sets needed to cover $\mathbb{R}^2$.



$\chi_m(G_{\{1\}}) \leq 7$

Density

$|\cdot|$ – Lebesgue measure

Definition

Density of A on domain Ω is

$$d_\Omega(A) = \frac{|A \cap \Omega|}{|\Omega|}$$

$Q(x, r)$ – square centered at x of side length r

Definition

Density of A is

$$d(A) = \lim_{R\to\infty} d_{Q(x,R)}(A)$$

Density decay

Definition

$$m(D) = \max_{A \text{ is } D\text{-avoiding}} d(A)$$

is the maximum density of a D-avoiding set.

Theorem

For any finite sets D_1 and D_2

$$\lim_{t \to \infty} m(D_1 \cup t \cdot D_2) = m(D_1) m(D_2)$$

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Corollary (Furstenberg-Katznelson-Weiss,Falconer-Marstrand,Bourgain)

If $d(A) > 0$ then all sufficiently large distances occur between points of A.

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Proof.

- Let $d_1, d_2, \ldots \notin \mathrm{dist}(A)$ grow sufficiently fast.
- Let $D_i = \{d_1, \ldots, d_i\} = D_{i-1} \cup d_i \cdot \{1\}$.
- $m(D_i) \leq m(D_{i-1})m(\{1\}) + o(1)$.
- Then $d(A) \leq m(D_i) \leq m(\{1\})^i + o(1)$. □

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- Then $d(A) \leq m(D_i) \leq m(\{1\})^i + o(1)$.
- ... and $\chi_m(G_{D_i}) \geq (1/m(\{1\}))^i$

Chromatic number is...

Corollary

For any k there is a set of k distances D such that

$$\chi_m(G_D) \geq 3^k$$

Observation

For any set of k distances D

$$\chi(G_D) \leq \chi_m(G_D) \leq 7^k$$

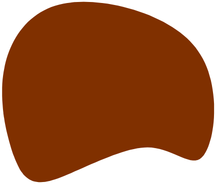
Theorem

There is a set of k distances D such that

$$\chi(G_D) \geq k\sqrt{\log k}$$

Main idea

Does A avoid distance d?



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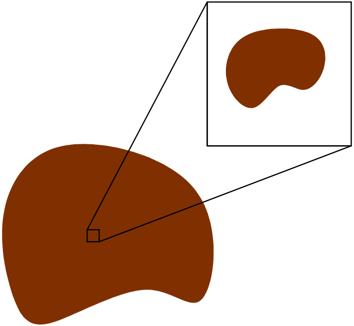

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$I(A) = \int_{\mathbb{R}^2} \mathbf{1}_A(x) \int_{S^1} \mathbf{1}_A(x+y)\, dy\, dx$ – counts points at distance 1

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$$I(A) = \int_{\mathbb{R}^2} \mathbf{1}_A(x) \int_{S^1} \mathbf{1}_A(x+y)\,dy\,dx$$ – counts points at distance 1

Crucial observation:

$$I(A) = \int_{\mathbb{R}^2} \mathbf{1}_A(x) \int_{\mathbb{R}^2} \mathbf{1}_A(x+y)\,d\sigma(y)\,dx \qquad \sigma \text{ – arclength on } S^1$$

$$= \int_{\mathbb{R}^2} |\widehat{\mathbf{1}_A}(\xi)|^2 \underbrace{\hat{\sigma}(\xi)}_{\text{Decays to zero as } |\xi| \to \infty} d\xi$$

High-frequency (small-scale) details do not contribute much.

Dessert

Definition

A D-avoiding set A is *locally optimal* if for no Ω of finite measure there is a D-avoiding A' such that

$$A' \setminus \Omega = A \setminus \Omega$$
$$|A' \cap \Omega| > |A \cap \Omega|$$

Dessert theorem

For any finite D a locally optimal D-avoiding set exists.