Extremal Combinatorics: homework #1[*]
Due 31 January 2026, at 10pm

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

1. Prove that every 3-uniform hypergraph with $\varepsilon\binom{n}{3}$ edges contains an independent set of size at least $\frac{1}{10}\varepsilon^{-1/2}$.

2. Show that there are constants $c = c(s,t)$ and $C = C(s,t)$ such that whenever G is a graph with $\varepsilon\binom{n}{2}$ edges, where $\varepsilon > Cn^{-1/s}$ edges, then G must contain at least $c\varepsilon^{st}n^{s+t}$ copies of $K_{s,t}$.

3. Greedy Greta has an idea for a new proof of Turán's theorem: to locate a K_r inside a graph G with $(1-\varepsilon)\binom{n}{2}$ edges, pick any vertex v of largest degree, and then find K_{r-1} inside $N(v)$ inductively. Does her idea work? If not, can she, using this method, still find a copy of K_r with $r = r(\varepsilon)$ that tends to infinity as $\varepsilon \to 0$?

4. Prove that there exists a constant $C > 0$ such that the following holds: Every n-vertex graph G with $\varepsilon\binom{n}{3}$ triangles contains a subset $U \subset V[G]$ of size at least $(1/\varepsilon)^C$ such that the induced subgraph $G[U]$ is triangle-free. What value of C do you obtain?

5. Show that every n-vertex triangle-free graph of minimum degree more than $2n/5$ is bipartite.

6. A *walk* of length ℓ in a graph G is a sequence of (not necessarily distinct) vertices $(v_0, v_1, \ldots, v_\ell)$ such that $v_i v_{i+1}$ is an edge for all i. Let G be an n-vertex graph of average degree d.

   (a) Prove that G contains at least nd^2 walks of length 2.

   (b) Prove that G contains at least nd^4 walks of length 4.

[*]This homework is from `http://www.borisbukh.org/ExtremalCombinatorics26/hw1.pdf`.

(c) Prove that G contains at least $\big(1-\varepsilon(d)\big)nd^4$ walks $(v_0, v_1, v_2, v_3, v_4)$ of length 4, all of whose vertices are distinct apart from possibly $v_1 = v_3$, where $\varepsilon(d) \to 0$ as $d \to \infty$. [Possible approach: delete low-degree vertices.]

(d) (Optional) Prove that G contains at least nd^3 walks of length 3.

7. Let p be an odd prime, and let $\mathbb{F}_{p^2}$ be a finite field with p^2 elements. Define the binary operation

$$a \circ b \stackrel{\text{def}}{=} \begin{cases} ab & \text{if } b \text{ is a square,} \\ a^p b & \text{if } b \text{ is not a square.} \end{cases}$$

Denote by P_0 the set of nonzero triples $(x, y, z) \in \mathbb{F}_{p^2}^3$, and let $\sim$ be the binary relation on P_0 defined by $(x, y, z) \sim (x \circ \lambda, y \circ \lambda, z \circ \lambda)$.

Let $P \stackrel{\text{def}}{=} P_0/\sim$ be the set of "points", and for each nonzero $(a, b, c) \in P_0$, define the "line" $\ell_{(a,b,c)} \stackrel{\text{def}}{=} \{(x, y, z) \in P : a \circ x + b \circ y + c \circ z = 0\}$. Prove that these definitions make sense, and the sets of points and lines thus defined form a projective plane.