Algebraic Structures: homework #2*
Due 30 January 2023, at 9am

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

1. Let A and B be subgroups of a group G. Show that $A \cup B$ is a subgroup of G if and only if $A \subseteq B$ or $B \subseteq A$.

2. Prove that $\mathrm{GL}_2(\mathbb{R})$ contains a proper subgroup G which in turn contains D_{2n} for every $n \geq 3$. [Here, 'proper' means that $G \neq \mathrm{GL}_2(\mathbb{R})$.]

3. For a nonnegative integer t, a t-cycle is a cycle of length t, i.e., an element of a symmetric group of the form $(a_1\, a_2\, \cdots\, a_t)$. Suppose σ and τ are both 3-cycles, and their product $\sigma\tau$ is a t-cycle, for some t. What are the possible values of t? [Your answer must consist of a set of possible values for t, examples showing that each of these values can occur, and a proof that no other values are possible.]

4. Let $X = \{1, 2, 3, \dots\}$ be the set of positive integers. Let S_X be the symmetric group on X.
   (a) Show that the group S_X is infinite.
   (b) Let $G = \{\pi \in S_X : \pi(i) = i$ for all but finitely many $i \in X\}$. Show that G is a subgroup of S_X.
   (c) (Not graded, just for fun) Can you show that S_X is uncountable, but G is countable?

5. Let G be a group. For the purpose of this problem, call an unordered pair of distinct elements $\{x, y\}$ in G *non-commuting* if $xy \neq yx$. Show that if G is non-abelian, then there are at least two distinct pairs of non-commuting elements. [Hint: start with a non-commuting pair $\{x, y\}$ and tweak it. Be careful to check that the new pair is not the same as $\{x, y\}$, e.g., the same two elements but written in a different order.]

*This homework is from `http://www.borisbukh.org/AlgebraicStructures23/hw2.pdf`.