Algebraic Structures: homework #12
Due 25 November 2024, at 9am via Gradescope

To receive full credit, all work must be shown. A passage means what careful but unimaginative reader thinks it does. Add details if in doubt. The problems should be written neatly and in order they were assigned.

A typical homework assignment is graded out of 20 points: 4 points for correctness of each problem. Bonus points result in additional credit.

0. (Ungraded)

   - Finish reading Section 5.3; this is what we covered by the 12th week. (We skipped Section 5.2.) Did you find any mistakes or typos? If you did not, you might not have read carefully enough.
   - Continue reading Chapter 5.

1. Problem 11 on page 215.

2. (a) Prove that, for every positive integer n, there exist polynomials $f_n, g_n \in J[x]$ such that $\cos nx = f_n(\cos x)$ and $\sin nx = g_n(\cos x)\sin x$.

   (b) Use part (a) to prove that $\sin 1^\circ$ and $\cos 1^\circ$ are algebraic numbers.

3. Let $f \in J[x]$ be a polynomial of odd degree. Prove that f is irreducible in $J[x]$ if and only if it is irreducible in $J[i][x]$. [Hint: Define and use conjugation for elements of $J[i][x]$.]

4. Suppose F is a field such that $2 \neq 0$ in F (i.e. characteristic of F is not 2, per definition on page 129). Assume that E is an extension of a field F satisfying $[E : F] = 2$. Show that a minimal polynomial over F of any element in $E \setminus F$ has two distinct roots in E.

5. Let p be a prime number.

   (a) Adapt Euclid's proof of infinitude of prime numbers to show the following: in the ring $J_p[x]$ there are infinitely many irreducible polynomials.

(b) Use the preceding part to deduce existence of arbitrarily large extensions of J_p that have only finitely many elements.

[For this problem you must follow the suggestions in the problem statement. Proofs using other ideas will not earn credit.]

6. (Bonus; 2 points) Let $f \in J[x]$ be an irreducible polynomial of degree $d \geq 2$, and suppose that $\alpha \in \mathbb{R}$ is its root.

   (a) Prove that there exists a constant $M > 0$ (that depends on f and α) such that $|\alpha - p/q| \geq \frac{M}{q^d}$ holds for all rational numbers p/q. [Hint: plug p/q into f and use a bit of calculus.]

   (b) Use part (a) to prove that the number

$$\alpha_0 \stackrel{\text{def}}{=} \frac{1}{10^{1!}} + \frac{1}{10^{2!}} + \frac{1}{10^{3!}} + \frac{1}{10^{4!}} + \cdots$$

   is transcendental.