Algebraic Structures: homework #10
Due 11 November 2024, at 9am via Gradescope

To receive full credit, all work must be shown. A passage means what careful but unimaginative reader thinks it does. Add details if in doubt. The problems should be written neatly and in order they were assigned.

A typical homework assignment is graded out of 20 points: 4 points for correctness of each problem. Bonus points result in additional credit.

0. (Ungraded)

   - Vote tomorrow, Tuesday, November 5th. It matters more than this class.
   - Finish reading Section 3.10; this is what we covered by the 10th week. Did you find any mistakes or typos? If you did not, you might not have read carefully enough.
   - Continue reading Chapter 3.

1. Suppose that integer D is not a perfect square. Let $N(a+b\sqrt{D}) = a^2 - b^2 D$ be the norm on $J[\sqrt{D}]$.

   (a) Prove that $x \in J[\sqrt{D}]$ is a unit if and only if $N(x) \in \{-1, +1\}$.

   (b) Find all units in $J[i]$.

   (c) Prove that $J[\sqrt{2}]$ contains infinitely many units. [Once you find three units, the rest are easy.]

2. Recall that we proved that a prime number p is a sum of two squares iff $p = 2$ or $p \equiv 1 \pmod 4$. In this exercise, we consider what happens when p is not prime. Let

$$S \overset{\text{def}}{=} \{a^2 + b^2 : a, b \in J\}.$$

   (a) Prove that $x, y \in S$ implies that $xy \in S$.

   (b) Let p be a prime number satisfying $p \equiv 3 \pmod 4$, and suppose that p^e is the largest power of p dividing positive integer $n \in S$. Prove that e is even.

(c) Use the preceding two parts to describe necessary and sufficient condition on a prime factorization of a positive integer n to be an element of S.

[Factoring in $J[i]$ helps.]

3. Problem 3 on page 158.

4. Problem 6 on page 158.

5. Problem 4 on page 161.