Algebraic Structures: homework #1
Due 3 September 2024, at 9am via Gradescope

To receive full credit, all work must be shown. A passage means what careful but unimaginative reader thinks it does. Add details if in doubt. The problems should be written neatly and in order they were assigned.

A typical homework assignment is graded out of 20 points: 4 points for correctness of each problem. Bonus points result in additional credit.

0. (Ungraded)
   - Read Chapter 1 of the book to familiarize yourself with the notation used in the book, and recall some important concepts from Concepts.
   - Start reading Chapter 2.

1. Suppose that G is a group such that $a = a^{-1}$ holds for every $a \in G$. Show that G is abelian.

2. Prove the claim on p.29 of the book that the "usual rules of exponents prevail" in groups, i.e., for all integer m and n
$$a^m \cdot a^n = a^{m+n}, \tag{1}$$
$$(a^m)^n = a^{mn}. \tag{2}$$
   Hint: you will need to consider different cases depending on whether m and n are positive, negative or zero.

3. Problem 14 on page 36, with the extra assumption that G is non-empty.

4. Problem 17 on page 36.

5. Let $G = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$.
   (a) Prove that G is a group under the usual addition.
   (b) Prove that $G \setminus \{0\}$ is a group under the usual multiplication. [Hint: multiply both numerator and denominator by the same quantity.]

6. (Bonus; 2 points) Problem 26 on page 37.