Algebraic Structures: homework #5*
Due 8 March 2021 at 4:15pm

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

1. Let $n \geq 2$. Show that A_n contains a subgroup isomorphic to S_{n-2}.

2. Let $n \geq 5$. Suppose that H is a subgroup of A_n isomorphic to S_{n-1}.

   (a) Explain how to use an action of A_n on left cosets of H to construct a homomorphism $A_n \to S_{n/2}$.

   (b) Prove that the homomorphism from (a) is injective, and deduce that A_n contains no subgroup isomorphic to S_{n-1}. [Hint: you need to use that $n \geq 5$.]

3. Suppose H is a subgroup of a group G. Show that $K = \bigcap_{g \in G} gHg^{-1}$ is a normal subgroup of G and that it is the largest normal subgroup contained in H (i.e., every normal subgroup of G contained in H is contained in K).

4. Let G be a finite group, and consider the action of G on itself by left multiplication. Let $\pi\colon G \to S_G$ be the corresponding homomorphism. Show that if $x \in G$ is an element of order n and $|G| = mn$, then $\pi(x)$ is a product of m n-cycles.

5. Generalize any one problem from homeworks #1 through #4. You must say which problem you are generalizing, state your generalization, and provide a solution to that generalization. [Saying that "X is a generalization of Y" means that X implies Y. You do not need to prove that your generalization is indeed a generalization, but it must be a strict generalization, i.e., $X = Y$ is not allowed.]

*This homework is from `http://www.borisbukh.org/AlgebraicStructures21/hw5.pdf`.