Algebraic Methods in Combinatorics: homework #1*
Due 4 February 2019, at start of class

Collaboration and use of external sources are permitted, but discouraged, and must be fully acknowledged and cited. Collaboration may involve only discussion; all the writing must be done individually.

The number of points for each problem is specified in brackets. The problems appear in no special order.

1. [2] Suppose $\mathcal{S}$ and $\mathcal{T}$ are two families of subsets of $[n]$ such that $|S \cap T|$ is odd for every $S \in \mathcal{S}$ and $T \in \mathcal{T}$. Show that $|\mathcal{S}||\mathcal{T}| \leq 2^{n-1}$.

2. Not far from Eventown and Oddtown there lies another settlement: Squaretown. It has n inhabitants, who have adopted for an even-stranger city charter than its sister towns. In Squaretown, each club has square-many members, and any two clubs share square-many members.

   (a) [1] Show that Squaretown can have no more than $2^{O(\sqrt{n}\log n)}$ clubs.

   (b) [1] Show that Squaretown can have as many as $2^{\Omega(\sqrt{n})}$ clubs.

3. [1] Let $v_1, \ldots, v_m$ be vectors with n integer entries, each of which is 0 or 1. Show that they are linearly independent over $\mathbb{Q}$ if and only if they are linearly independent over $\mathcal{F}_p$ for all sufficiently large p. How large is "sufficiently large"?

4. [2] Let s be a positive integer. Suppose $A_1, \ldots, A_m$ are subsets of an n-element set such that the sizes of $A_1, \ldots, A_m$ are not divisible by s, but $|A_i \cap A_j|$ are, for any distinct i and j. For $s = 6$, show that $m \leq 2n$. For general s, show that there is a constant $c(s)$ such that $m \leq c(s)n$. (Open problem: must $c(s)$ depend on s?)

*This homework is from `http://www.borisbukh.org/AlgMethods19/hw1.pdf`.

5. (a) [1] Show that for each B there exists a number $\varepsilon > 0$ such that if $f(x)$ is a non-zero univariate polynomial whose coefficients are integers not exceeding B, then $f(x) \neq 0$ for all $x \in (0, \varepsilon)$.

   (b) [1] Show that there exists $r_0 = r_0(d)$ sufficiently large so that the following holds for all $r > r_0$. Whenever $P \subset \mathbb{R}^d$ is a two-distance set in which distance between any two points are 1 and r, then $|P| \leq d + 1$. (Hint: prove the same for one-distance sets first.)

6. [2] Let $A_1, \ldots, A_m$ be subsets of an n-element set. Suppose $|A_i|$ is odd for all i, and all triplewise intersections $|A_i \cap A_j \cap A_k|$ are even (for distinct i, j and k). Show that there is a constant C such that $m = O(n^C)$.